Bidirectional Dijkstra's Algorithm is Instance-Optimal

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SOSA 2025

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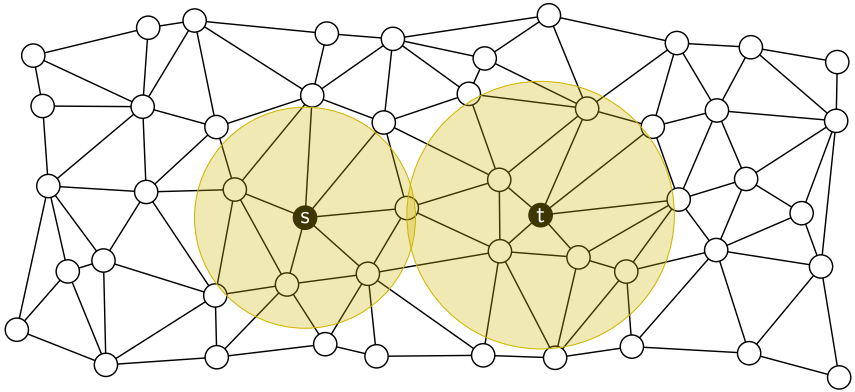
Intro



This talk: bidirectional search is optimal on every graph

How quickly can you find the shortest path?

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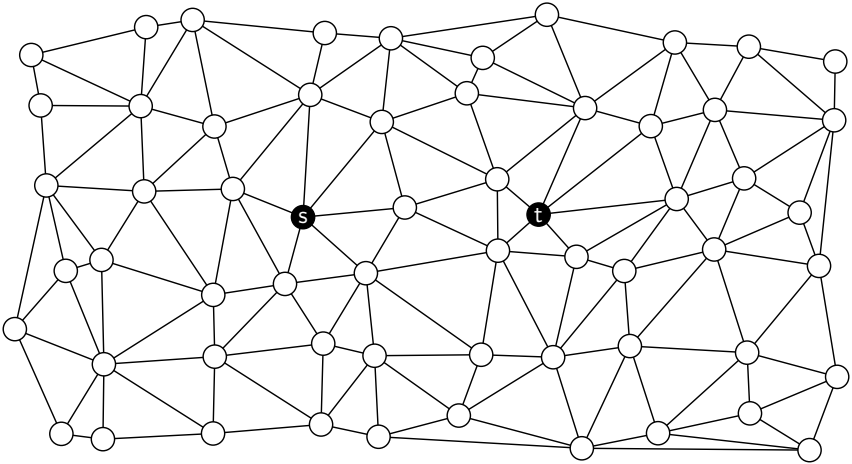
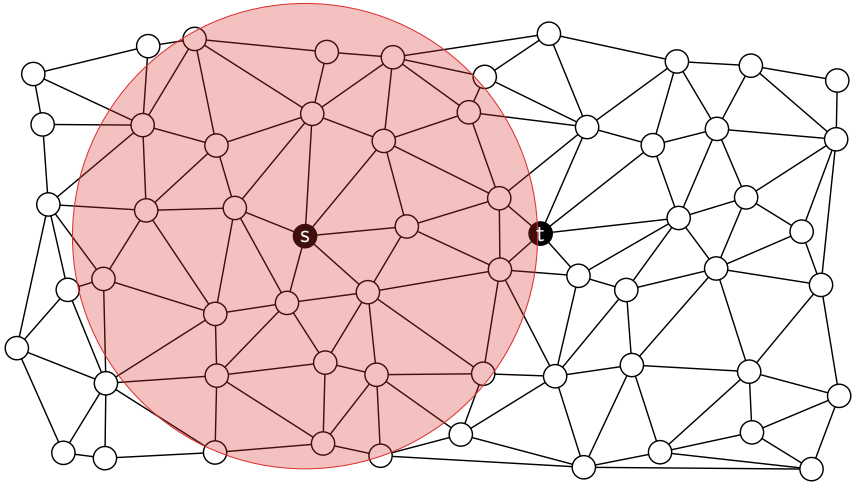


How quickly can you find the shortest path: Dijkstra



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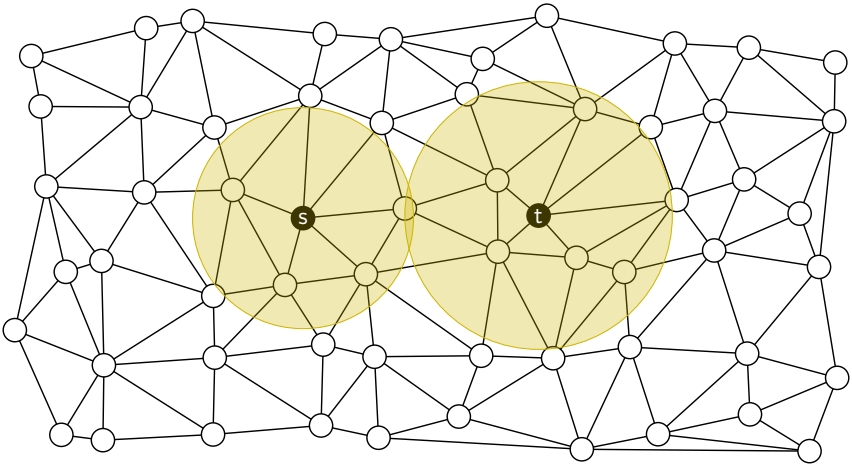

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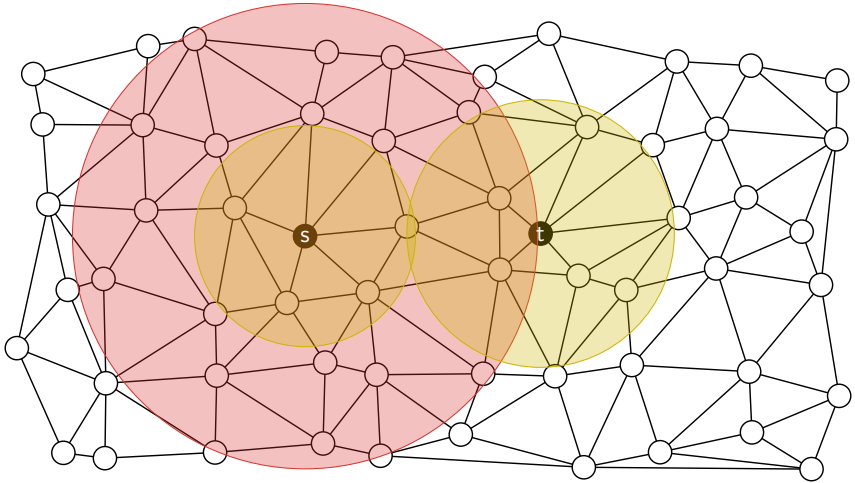
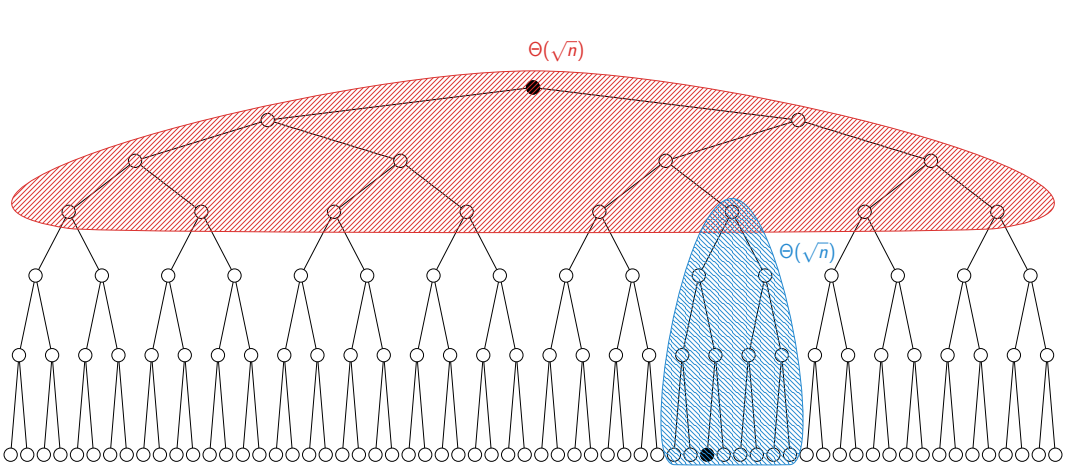


Bidirectional search can make a big difference!



Outline

- problem statement
- model
- instance-optimality
- the result
- sketch of the key idea
- related work & debrief

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- **Neighbor**$(v, i) \to$ (vertex id, edge weight) of i-th neighbor

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(results also hold for time complexity up to a $\log n$ factor)

Instance Optimality

A is instance-optimal if:

$$\exists c \; \forall \text{correct } A' \; \forall input \quad \text{Queries}_A(input) \leq c \cdot \text{Queries}_{A'}(input)$$

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Our result: A variant of bidirectional Dijkstra is instance-optimal (w. r. t. query complexity)

Proof: Key (but insufficient) idea

Claim: Let $uv \in E$ be such that $d(s,u) + d(v,t) < d(s,t)$. Then any correct algorithm needs to see uv.

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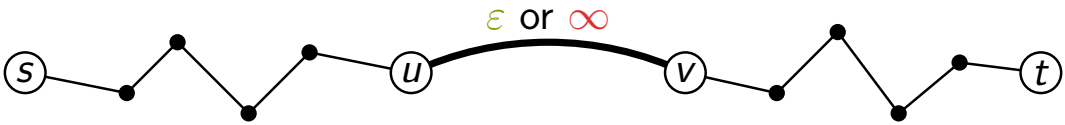
Claim: Let $uv \in E$ be such that $d(s,u) + d(v,t) < d(s,t)$. Then any correct algorithm needs to see uv.
Proof: Otherwise it cannot distinguish between $\ell(uv) = \varepsilon$ and $\ell(uv) = \infty$ and is incorrect in at least one case.



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. . . but with a stronger claim, we can make it work

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- previous optimality results [Eck+17; Sha+19] assume some knowledge of the graph and are weaker than instance-optimality

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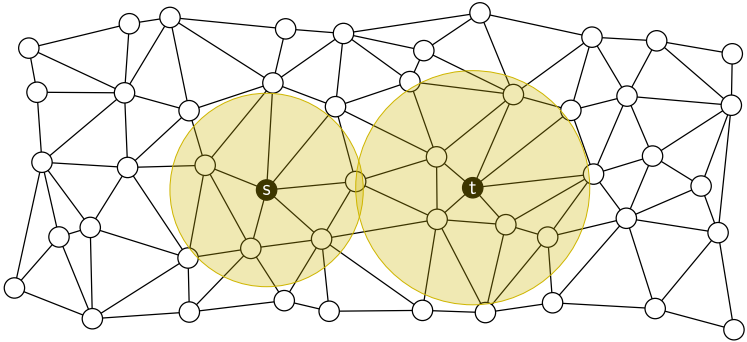
- previous optimality results [Eck+17; Sha+19] assume some knowledge of the graph and are weaker than instance-optimality
- bidirectional search has many variants [Dan63; Nic66; Dre69; Poh69], none of the previous variants is instance-optimal
  - we design an instance-optimal variant
- we also prove that:
  - allow zero-length edges $\Rightarrow$ instance-optimality impossible
  - **un**weighted graphs $\Rightarrow$ instance-optimality only up to a max-degree factor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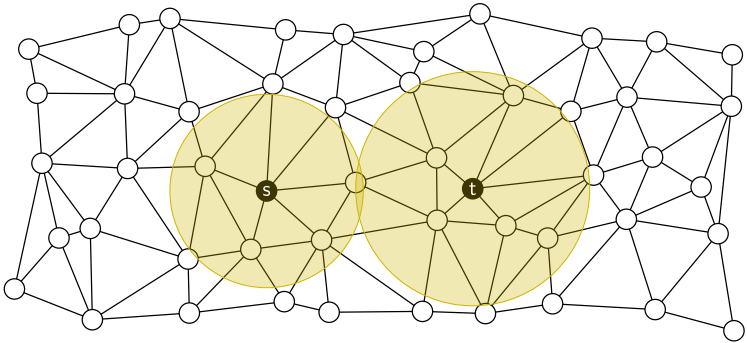

Thank you!

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Backup Slides

With zero-length edges, instance-optimality is impossible

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Intuition:

- fix G, s, t
- set all lengths to 0
- now finding the shortest st-path is equivalent to finding *any* st-path
- there is a set of trivial dumb algorithms that guess what the path is

Banning 0-length edges helps because when a dumb algorithm finds a path, it still has to check if there isn't some shorter one.

The Bidirectional Search Meta-Algorithm

while the *stopping condition* is not met:
  use a *selection rule* to pick execution $\in \{\text{forward}, \text{backward}\}$
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    - not instance-optimal!
  - **our rule:** alternate doing $\mathcal{O}(1)$ work in the forward and backward iterations