Fast and Simple Sorting Using Partial Information

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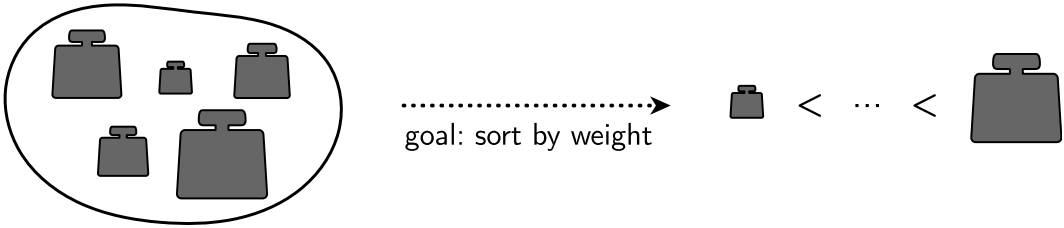
This talk: resolving a sorting problem from 1976

Sorting Under Partial Information (SUPI)



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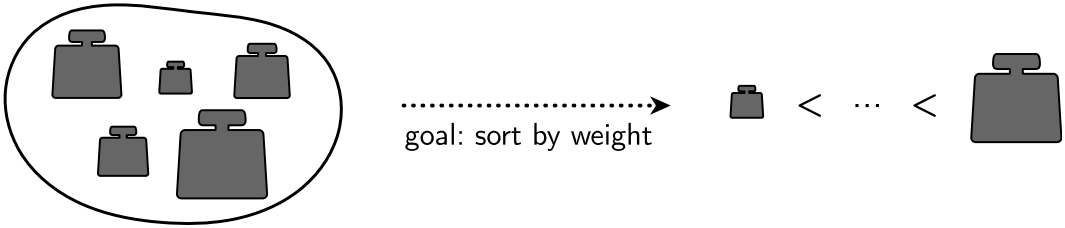

$x < y$

1. I can do my own comparisons

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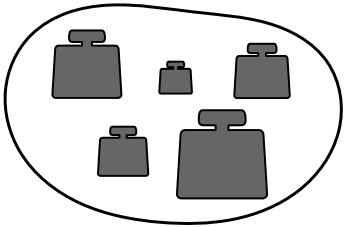
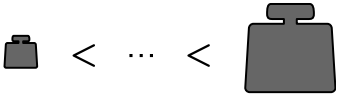


goal: sort by weight



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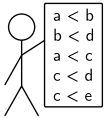

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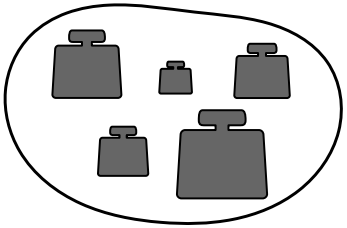

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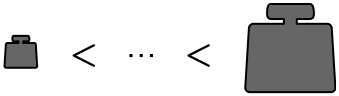

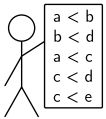

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Goal: Find the sorted order. Optimize for 1) #comparisons 2) time.

Outline

- What's known
  - problem properties
    - when is the problem interesting?
  - lower bound
  - related work

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- What's known
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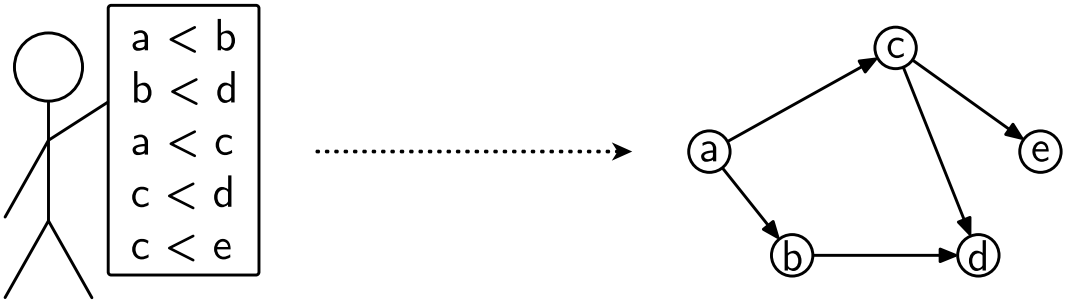
    - when is the problem interesting?
  - lower bound
  - related work
- What's new
  - our optimal algorithm
  - relationship with heaps
  - connections to other results

Part I: What's known

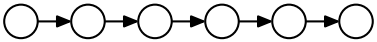
Graph rephrasing



Promise: $\forall uv \in E, \quad u < v$

(= the comparison list isn't fake)

Lower Bound



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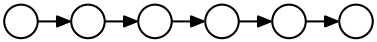

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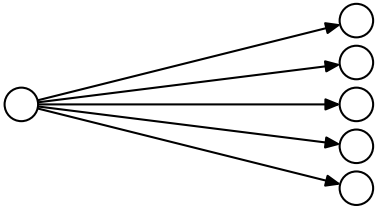

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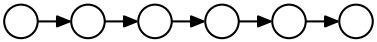

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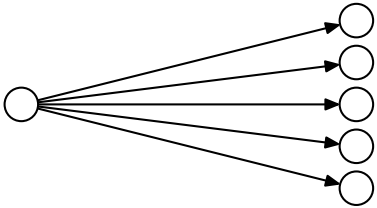

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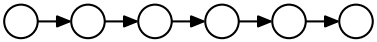

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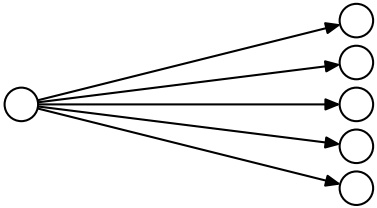

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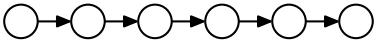

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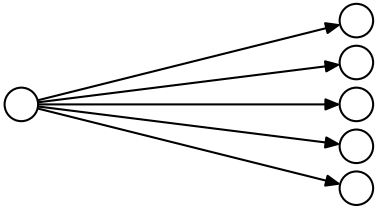

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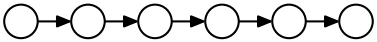

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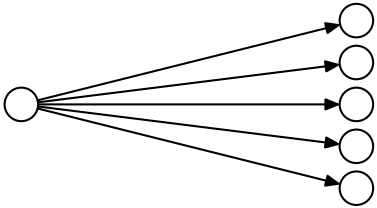

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Related Work

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Our approach: ✨ just put it in a heap ✨

Part II: What's new

Beyond-worst-case heaps

Motivation: Can comparison-based heaps / search trees break the $\mathcal{O}(\log n)$ barrier?

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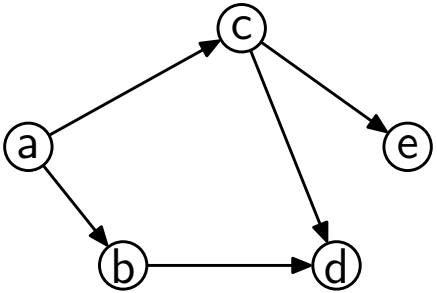
- much faster than traditional heaps if the heap is large, but most ExtractMins are of "fresh" items

Our Algorithm: Topological Heapsort



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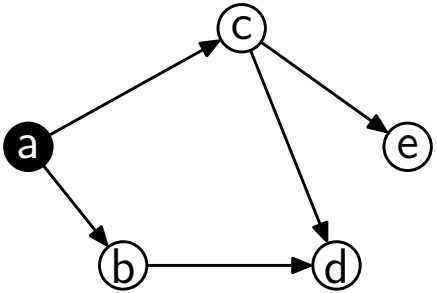

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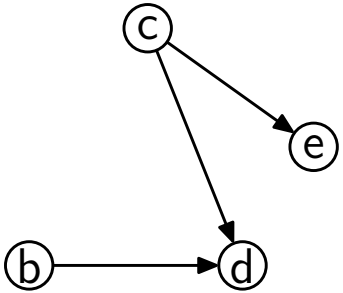

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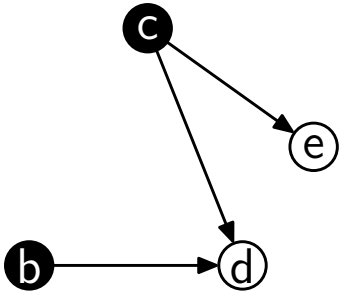

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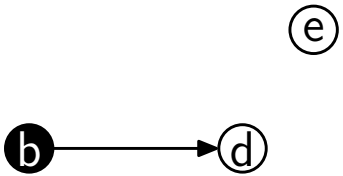

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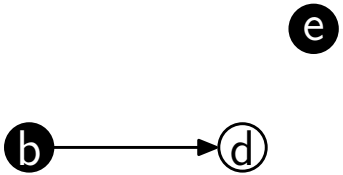

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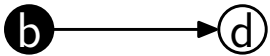

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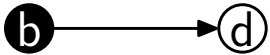

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Main result: this algo is optimal w.r.t. time if we use a working-set heap.
. . . and optimal w.r.t. #comparisons after a slight tweak.

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Connection 1: Dijkstra

Dijkstra (with a working-set heap) is optimal on every graph [Hae+24].

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- Similar proof techniques for optimality

Connection 2: Follow-up work [SOSA'25]

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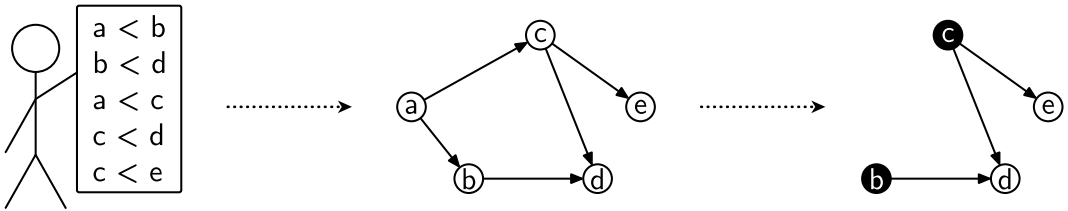
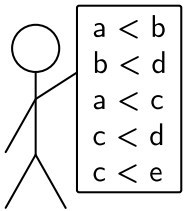
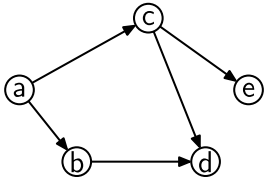
Come check it out tomorrow!

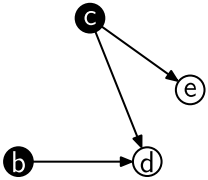
Conclusion



Conclusion







Thank you!

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