Near-Universally-Optimal Differentially Private Minimum Spanning Trees

Richard Hladík[1,2], Jakub Tětek[2]

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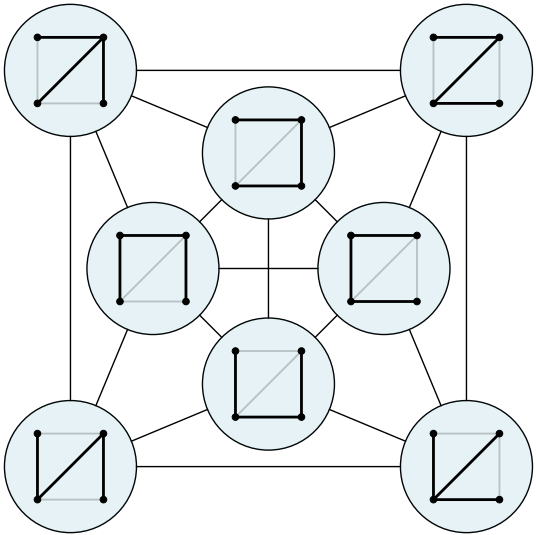
Intro

This talk: differential privacy bounds via combinatorial properties of spanning trees



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- **goal:** privately release the (approximate) MST of G
  - not just its weight (that's easier), release the spanning tree itself
  - can't just release a MST (not private), instead release a random spanning tree of G that's "close to minimal" in expectation

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  - public unweighted connected graph G
  - private real edge weights $\mathbf{w}$

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  - **approximate MST**: *the expected weight is small*

$$\text{additive error} = \mathbb{E}_{T \sim A(G,\mathbf{w})}[\mathbf{w}(T)] - \mathbf{w}(\underbrace{T^*}_{\text{MST of } G}) \quad \text{is small}$$

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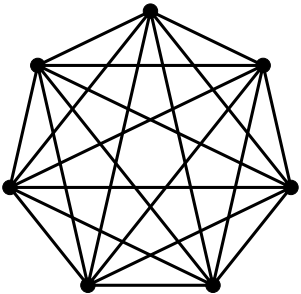
Our result 3	$\exists G \quad \Omega(n \log n/\varepsilon)$	special case; implies [Sea16] is worst-case tight

Why care about D_G?



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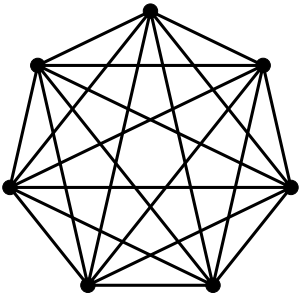

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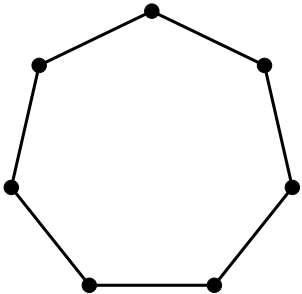

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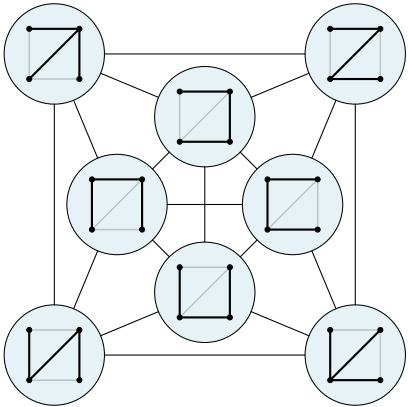
Defining D_G

- $\mathcal{T}$ = space of spanning trees of G



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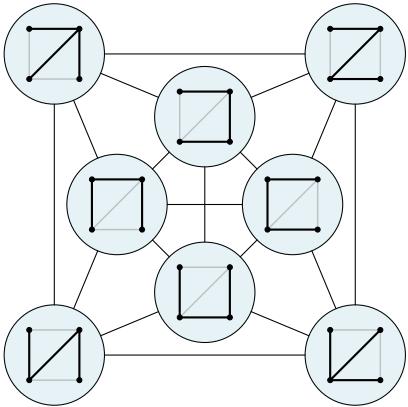

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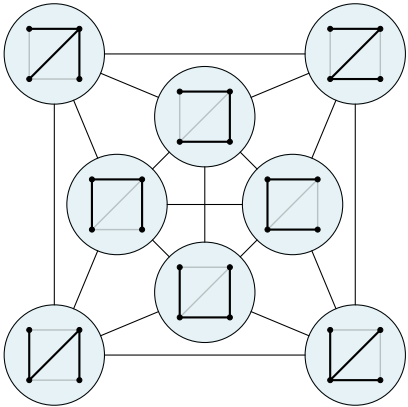

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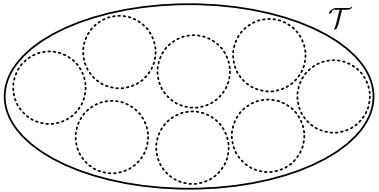
High-level proof summary:

- use well-known packing lower bounds [Vad17]
- lower bound reduces to fitting many balls into the space of spanning trees



- do that via error-correcting codes

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Observation: This problem is reducible to private MST: (up to the error scaling by a $\frac{1}{2}$ factor)

- given $T_x \in \mathcal{T}$, define $\mathbf{w}_x(e) = 0$ if $e \in T_x$, $\frac{1}{2}$ otherwise.
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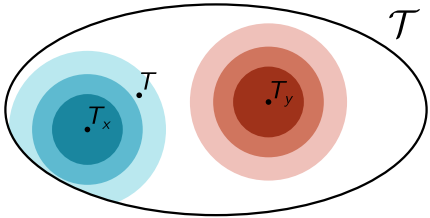
All we need is a lower bound for the private spanning tree problem.

Lower bound 2/3: reduction to packing (intuition)



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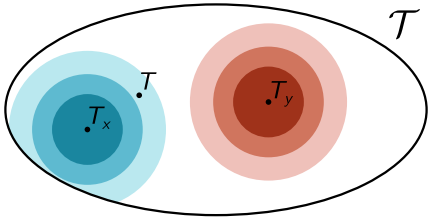

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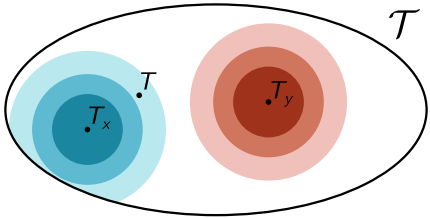

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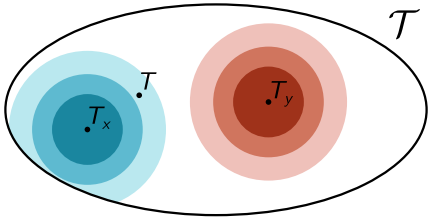

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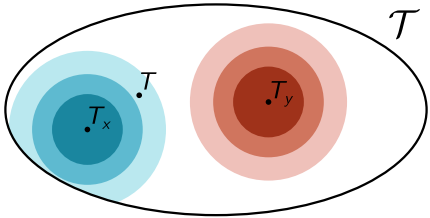

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- if we find enough pairwise distant $T_1, \ldots, T_k$, we get a lower bound

Lower bound 3/3: finding a good packing

Theorem (simplified): Let $\mathcal{C} \subseteq \mathcal{T}$ satisfy $d(T_x, T_y) \geq D_G/10$ for all distinct $T_x, T_y \in \mathcal{C}$. Let A be a ε-DP mechanism for MST. Then $\exists \mathbf{w}$ such that $A(G, \mathbf{w})$ has $\Omega(\log |\mathcal{C}|/\varepsilon) - \mathcal{O}(1)$ expected error.

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$$\Rightarrow \Omega(n \log n/\varepsilon) - \mathcal{O}(1) \text{ worst-case l.b. } (\exists G)$$
$$\& \quad \Omega(D_G/\varepsilon) - \mathcal{O}(1) \text{ universal l.b. } (\forall G)$$

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